

Spin-rotation couplings: spinning test particles and Dirac field

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Dedicated to Bahram Mashhoon for his 60th birthday

Abstract

The hypothesis of coupling between spin and rotation introduced long ago by Mashhoon is examined in the context of “1+3” and “3+1” space-time splitting techniques, either in special or in general relativity. Its content is discussed in terms of classical (Mathisson-Papapetrou-Dixon-Souriau model) as well as quantum physics (Foldy-Wouthuysen transformation for the Dirac field in an external field), reviewing and discussing all the relevant theoretical literature concerning the existence of such effect. Some original contributions are also included.

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1 Introduction

There is a huge literature on the problem of the existence of spin-rotation couplings either in accelerated frames in Minkowski space-time or in linearized gravity, due to the absence of experimental signatures.

Mashhoon was a pioneer in the investigation of the various aspects of this topic emphasizing the relevance of the *locality hypothesis* in special relativity

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and of the *equivalence principle* in the comparison of the acceleration and gravity effects (see e.g. [1, 2, 3, 4, 5] for reviews and an exhaustive collection of references).

The problem of the spin-rotation couplings is mainly formulated in the context of space-time splitting techniques, namely within the two different (and competing) points of view termed “1+3” and “3+1” splittings.

In the first case (“1+3” *point of view*), the description of special and general relativistic effects is done by a time-like observer with an assumed known world line. As a consequence, the instantaneous 3-space of the observer is identified with the tangent space of vectors orthogonal to the observer 4-velocity at each instant of the observer proper time and usually it is coordinatized with Fermi-like 3-coordinates. In this way only local information accessible to the observer is used and this is considered physically acceptable.

However, this description holds only close the observer world line, because the various instantaneous 3-spaces will intersect each other at a distance from the world line of the order of the acceleration lengths [3, 5]. This means that there is neither a *globally* defined clock synchronization convention (replacing Einstein’s $\frac{1}{2}$ one, valid in special relativistic inertial frames) nor the possibility to formulate a well-posed Cauchy problem for Maxwell (or Yang-Mills, or Einstein) equations. Also the type of radar coordinates introduced in Ref.[6], based on Einstein convention of clock synchronization with light signals, have been shown to have similar limitations[7]. Moreover, within this point of view to have a consistent definition of instantaneous 3-spaces with an atlas of coordinates such that the Fermi coordinates are a local chart around the observer world line is still an open question.

The approach more suited to solve these problems is the “3+1” *point of view*, in which one gives a “3+1” splitting of Minkowski space-time, namely a nice foliation with space-like leaves, besides the observer world line; that is one defines a *global non-inertial frame centered on the observer*. Such a splitting is a *generalized clock synchronization convention*: each space-like leaf is an instantaneous 3-space (in general a curved Riemannian 3-manifold), which can be used as a Cauchy surface for field equations. To avoid coordinate singularities like the ones appearing either with Fermi coordinates or with rotating frames the foliation has to satisfy the *Møller admissibility conditions* [8] and its leaves must tend to space-like hyper-planes at spatial infinity. Then a global non-inertial frame centered on the given time-like observer can be built by defining generalized (observer-dependent and Lorentz scalar) radar 4-coordinates (τ, σ^r) : the time variable τ , labeling the simultaneity leaves Σ_τ , is an arbitrary monotonically increasing function of the observer proper time; σ^r are curvilinear 3-coordinates on each Σ_τ having the world line as origin ¹. If x^μ are Cartesian 4-coordinates in an inertial frame in Minkowski space-time, the coordinate transformation $x^\mu \mapsto \sigma^A = (\tau, \sigma^r)$ has an inverse $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ defining the embeddings

¹Often we shall use the notation $\vec{\sigma} = \{\sigma^r\}$ for the sake of simplicity. Moreover, we use a 4-metric with signature $\epsilon(+---)$, with $\epsilon = \pm$ according to both standard conventions.

$z^\mu(\tau, \sigma^r)$ of the simultaneity 3-surfaces. Using the notation

$$z_A^\mu = \partial z^\mu / \partial \sigma^A ,$$

the induced 4-metric is

$$g_{AB}(\tau, \sigma^r) = \eta_{\mu\nu} z_A^\mu(\tau, \sigma^r) z_B^\nu(\tau, \sigma^r)$$

and Møller admissibility conditions are given by

$$\epsilon g_{\tau\tau}(\tau, \sigma^u) > 0, \quad \epsilon g_{rr}(\tau, \sigma^u) < 0, \quad \epsilon \det [g_{rs}(\tau, \sigma^u)] < 0$$

$$\det \begin{vmatrix} g_{rr}(\tau, \sigma^u) & g_{rs}(\tau, \sigma^u) \\ g_{sr}(\tau, \sigma^u) & g_{ss}(\tau, \sigma^u) \end{vmatrix} > 0, \quad \forall r, s \text{ fixed}, \quad (1)$$

implying $\det [g_{AB}(\tau, \sigma^u)] < 0$.

In Ref.[9] it is shown that with each admissible “3+1” splitting are associated two congruences of time-like observers (the natural ones for the given notion of simultaneity):

- i) the Eulerian observers, whose unit 4-velocity field is the field of unit normals to the simultaneity surfaces Σ_τ ;
- ii) the observers whose unit 4-velocity field is proportional to the evolution vector field of components $\partial z^\mu(\tau, \sigma^r)/\partial \tau$: in general this congruence is non-surface forming having a non-vanishing vorticity.

Both the “3+1” and the “1+3” point of view may appear not so physical², but it allows to arrive at a well-posed Cauchy problem for field equations, i.e. to a mathematical control of *determinism* once the gauge freedom of the given field theory has been fixed. Moreover, it allows to formulate an action principle (*parametrized Minkowski theories*) for every special relativistic isolated system (particles, strings, fields, fluids) for which a Lagrangian description is known, such that the transition from an admissible “3+1” splitting (with associated radar 4-coordinates) to another one (with new radar 4-coordinates) is obtained using a *gauge transformation*. This property is a consequence of the invariance of the action under frame-preserving diffeomorphisms and implies that the physics is independent from the choice of the synchronization convention, as expected. The same “3+1” point of view is the starting point of the canonical formulation of metric and tetrad gravity in globally hyperbolic asymptotically flat (with suitable boundary conditions at spatial infinity) space-times, where there is a notion of global time and where general covariance (invariance under diffeomorphisms) again implies the gauge equivalence of the admissible “3+1” splittings. i.e. of the global non-inertial frames (the only ones allowed by the *equivalence principle*).

In addition, differently from the Fermi coordinates, it is possible to give an *operational* definition of the generalized radar 4-coordinates. As shown in

²Actually, the “3+1” approach requires the knowledge of the data on a whole space-like hyper-surface which is not factual; similarly the “1+3” is not factual because it requires the knowledge of the data on a whole world line, i.e. also in the “future.”

Refs.[9, 10], given four functions satisfying certain restrictions due to Møller conditions, the on-board computer of a spacecraft may use them to build a grid of radar 4-coordinates in its future. All these properties are explained in detail in Refs.[9, 10, 11] for special relativity and in Refs.[11, 12] for general relativity. Moreover, in special relativity the restriction of parametrized Minkowski theories to inertial frames allowed to find the *inertial-rest-frame Wigner-covariant instant form of dynamics*[11, 13]. These results are possible due to a systematic use of Dirac theory of constraints in the Hamiltonian description of relativistic systems.

In this paper we mainly reconsider the problem of the spin-rotation couplings both from the “3+1” and “1+3” points of view, trying to clarify their interconnections.

2 Spin-Rotation Couplings in Special Relativity

2.1 Parametrized Minkowski Theories and the Rest-Frame Instant Form of Dynamics.

Parametrized Minkowski theories[11, 13] have been developed to describe isolated physical systems in non-inertial frames in such a way that different conventions for clock synchronization are connected by gauge transformations.

Given any isolated system admitting a Lagrangian description, one makes the coupling of the system to an external gravitational field and then replaces the 4-metric ${}^4g_{\mu\nu}(x)$ with the induced metric ${}^4g_{AB}[z(\tau, \sigma^r)]$ associated with an arbitrary admissible “3+1” splitting. The Lagrangian now depends not only on the matter configurational variables but also on the embedding variables $z^\mu(\tau, \sigma^r)$ (whose conjugate canonical momenta are denoted $\rho_\mu(\tau, \sigma^r)$). Since the action principle turns out to be invariant under frame-preserving diffeomorphisms, at the Hamiltonian level there are four first-class constraints

$$\mathcal{H}_\mu(\tau, \sigma^r) = \rho_\mu(\tau, \sigma^r) - l_\mu(\tau, \sigma^r) T^{\tau\tau}(\tau, \sigma^r) - z_s^\mu(\tau, \sigma^r) T^{\tau s}(\tau, \sigma^r) \approx 0$$

in strong involution with respect to Poisson brackets

$$\{\mathcal{H}_\mu(\tau, \sigma^r), \mathcal{H}_\nu(\tau, \sigma_1^r)\} = 0.$$

Here $l_\mu(\tau, \sigma^r)$ are the covariant components of the unit normal to Σ_τ , while $z_s^\mu(\tau, \sigma^r)$ are the components of three independent vectors tangent to Σ_τ . The quantities $T^{\tau\tau}$ and $T^{\tau s}$ are the components of the energy-momentum tensor of the matter distributed on Σ_τ , describing its energy and momentum densities. As a consequence, Dirac’s theory of constraints implies that the configuration variables $z^\mu(\tau, \sigma^r)$ are arbitrary *gauge variables*. Therefore, all the admissible “3+1” splittings, namely all the admissible conventions for clock synchronization, and all the admissible non-inertial frames centered on time-like observers are *gauge equivalent*. By adding four gauge-fixing constraints

$$\chi^\mu(\tau, \sigma^r) = z^\mu(\tau, \sigma^r) - z_M^\mu(\tau, \sigma^r) \approx 0$$

($z_M^\mu(\tau, \sigma^r)$ being an admissible embedding), satisfying the orbit condition

$$\det |\{\chi^\mu(\tau, \sigma^r), \mathcal{H}_\nu(\tau, \sigma_1^r)\}| \neq 0,$$

we identify the description of the system in the associated non-inertial frame centered on some given time-like observer chosen as origin. The resulting effective Hamiltonian for the τ -evolution turns out to contain the potentials of the *relativistic inertial forces* present in the given non-inertial frame. Since a non-inertial frame means the use of its radar coordinates, we see that already in special relativity *non-inertial Hamiltonians are coordinate-dependent quantities* like the notion of energy density in general relativity. As a consequence, the gauge variables $z^\mu(\tau, \sigma^r)$ describe the *spatio-temporal appearances* of the phenomena in non-inertial frames.

Inertial frames centered on inertial observers are a special case of gauge fixing in parametrized Minkowski theories, where the embeddings $z^\mu(\tau, \sigma^r)$ are linear in the radar 4-coordinates.

For each configuration of an isolated system there is a special “3+1” splitting associated with it: the foliation with space-like hyper-planes orthogonal to the conserved time-like 4-momentum of the isolated system.

This identifies an intrinsic inertial frame, the *rest-frame*, centered on a suitable inertial observer (the Fokker-Pryce center of inertia of the isolated system) and allows to define the *Wigner-covariant rest-frame instant form of dynamics* for every isolated system³. Its instantaneous 3-spaces Σ_τ are named *Wigner hyper-planes*, because 3-vectors lying into them transform as Wigner spin-1 3-vectors. See Refs. [14, 15] for the development of a coherent formalism describing all the aspects of relativistic kinematics for N particle systems, continuous bodies and fields generalizing all known non-relativistic results:

- i) the classification of the intrinsic notions of collective variables (canonical non-covariant center of mass; covariant non-canonical Fokker-Pryce center of inertia; non-covariant non-canonical Møller center of energy);
- ii) canonical bases of center-of-mass and relative variables;
- iii) canonical spin bases and dynamical body-frames for the rotational kinematics of deformable systems;
- iv) multipolar expansions for isolated systems and their open subsystems with a Hamiltonian formulation of the Mathisson-Papapetrou-Dixon-Souriau equations which puts control on their subsidiary conditions (see the first and third paper in Ref.[14]);
- v) the relativistic theory of orbits [15] (while the potentials appearing in the energy generator of the Poincare’ group determine the relative motion, the determination of the actual orbits in the given inertial frame is influenced by the potentials appearing in the Lorentz boosts);
- vi) the Møller radius (a classical unit of length identifying the region of non-covariance of the canonical center of mass of a spinning system around the covariant Fokker-Pryce center of inertia; it is an effect induced by the Lorentz

³This happens because in the gauge fixing use is made of the standard Wigner boost $L_\nu^\mu(p, \hat{p})$ ($p^\mu = L_\nu^\mu(p, \hat{p})\hat{p}^\nu$, $\hat{p}^\mu = \eta\sqrt{\epsilon p^2}(1; \vec{0})$, $\eta = \text{sign } p^0$).

signature of the 4-metric; it could be used as a physical ultraviolet cutoff in quantization).

vii) the definition of non-inertial rest frames, where the simultaneity leaves tend to space-like hyper-planes orthogonal to the total 4-momentum of the system at spatial infinity: only this family of embeddings is relevant for the “3+1” point of view of metric and tetrad gravity in globally hyperbolic space-times [11, 12].

Let us remark that in parametrized Minkowski theories a relativistic particle with world line $x_i^\mu(\tau)$ is described only by the 3-coordinates $\sigma^r = \eta_i^r(\tau)$ defined by $x_i^\mu(\tau) = z^\mu(\tau, \eta_i^r(\tau))$ and by the conjugate canonical momenta $\kappa_{ir}(\tau)$. The usual 4-momentum $p_{i\mu}(\tau)$ is a derived quantity satisfying the mass-shell constraint $\epsilon p_i^2 = m_i^2$ in the free case. Therefore, we have a different description for positive and negative energy particles. All the particles on an admissible surface Σ_τ are simultaneous by construction: *this eliminates the problem of relative times*, which for a long time has been an obstruction to the theory of relativistic bound states and to relativistic statistical mechanics (see Ref.[14, 15] and its bibliography for these problems and the related no-interaction theorem).

2.2 The Locality Hypothesis and Møller Conditions

Let us now consider a class of 4-coordinate transformations associated with the idea of accelerated observers as sequences of comoving observers (i.e. the locality hypothesis [9, 10]).

The admissible embeddings $x^\mu = z^\mu(\tau, \vec{\sigma})$, defined with respect to a given inertial system, must tend to parallel spacelike hyper-planes at spatial infinity. If $l^\mu = l_{(\infty)}^\mu \stackrel{def}{=} \epsilon_\tau^\mu [l_{(\infty)}^2 = \epsilon]$ is the asymptotic normal, let us define the asymptotic orthonormal tetrad ϵ_A^μ , $A = \tau, 1, 2, 3$, by using the standard Wigner boost for time-like Poincare’ orbits $L^\mu{}_\nu(l_{(\infty)}, \overset{\circ}{l}_{(\infty)})$ [$\overset{\circ}{l}_{(\infty)} = (1; \vec{0})$]:

$$\epsilon_A^\mu \stackrel{def}{=} L^\mu{}_\nu(l_{(\infty)}, \overset{\circ}{l}_{(\infty)})$$

with the property $\epsilon_A^\mu \eta_{\mu\nu} \epsilon_B^\nu = \eta_{AB}$ [$= \epsilon(+---)$]. Then a parametrization of the asymptotic hyper-planes is $z^\mu = x_o^\mu + \epsilon_A^\mu \sigma^A = x^\mu(\tau) + \epsilon_r^\mu \sigma^r$ with $x^\mu(\tau) = x_o^\mu + \epsilon_\tau^\mu \tau$ a time-like straight-line (an asymptotic inertial observer).

Let us define a family of “3+1” splittings of Minkowski space-time by means of the following embeddings

$$z^\mu(\tau, \vec{\sigma}) = x_o^\mu + \Lambda^\mu{}_\nu(\tau, \vec{\sigma}) \epsilon_A^\nu \sigma^A = \tilde{x}^\mu(\tau) + F^\mu(\tau, \vec{\sigma}), \quad F^\mu(\tau, \vec{0}) = 0,$$

$$\tilde{x}^\mu(\tau) = x_o^\mu + \Lambda^\mu{}_\nu(\tau, \vec{0}) \epsilon_\tau^\nu \tau, \tag{2}$$

$$F^\mu(\tau, \vec{\sigma}) = [\Lambda^\mu{}_\nu(\tau, \vec{\sigma}) - \Lambda^\mu{}_\nu(\tau, \vec{0})] \epsilon_\tau^\nu \tau + \Lambda^\mu{}_\nu(\tau, \vec{\sigma}) \epsilon_r^\nu \sigma^r,$$

$$\Lambda^\mu{}_\nu(\tau, \vec{\sigma}) \rightarrow_{|\vec{\sigma}| \rightarrow \infty} \delta_\nu^\mu, \quad \Rightarrow \quad z^\mu(\tau, \vec{\sigma}) \rightarrow_{|\vec{\sigma}| \rightarrow \infty} x_o^\mu + \epsilon_A^\mu \sigma^A = x^\mu(\tau) + \epsilon_r^\mu \sigma^r,$$

where $\Lambda^\mu{}_\nu(\tau, \vec{\sigma})$ are Lorentz transformations ($\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$) belonging to the component connected with the identity of $SO(3,1)$. While the functions $F^\mu(\tau, \vec{\sigma})$ determine the form of the simultaneity surfaces Σ_τ , the centroid $\tilde{x}^\mu(\tau)$, corresponding to an *arbitrary time-like observer* chosen as origin of the 3-coordinates on each Σ_τ , determines how these surfaces are packed in the foliation. Since the asymptotic foliation with parallel hyper-planes, having a constant vector field $l^\mu = \epsilon^\mu_\tau$ of normals, defines an inertial reference frame, we see that the foliation (3) with its associated non-inertial reference frame is obtained from the asymptotic inertial frame by means of *point-dependent Lorentz transformations*. As a consequence, the integral lines, i.e. the non-inertial Eulerian observers, origin of (non-rigid) non-inertial reference frames, are parametrized as a continuum of comoving inertial observers as required by the locality hypothesis.

Therefore, in the framework of parametrized Minkowski theories *the locality hypothesis can always be assumed valid modulo gauge transformations*.

An equivalent parametrization of the embeddings of this family of reference frames is

$$z^\mu(\tau, \vec{\sigma}) = x^\mu_o + \epsilon^\mu_B \Lambda^B{}_A(\tau, \vec{\sigma}) \sigma^A = x^\mu_o + U^\mu_A(\tau, \vec{\sigma}) \sigma^A = \tilde{x}^\mu(\tau) + F^\mu(\tau, \vec{\sigma}),$$

$$\tilde{x}^\mu(\tau) = x^\mu_o + U^\mu_\tau(\tau, \vec{0}) \tau,$$

$$F^\mu(\tau, \vec{\sigma}) = [U^\mu_\tau(\tau, \vec{\sigma}) - U^\mu_\tau(\tau, \vec{0})] \tau + U^\mu_r(\tau, \vec{\sigma}) \sigma^r, \quad (3)$$

where we have defined:

$$\Lambda^B{}_A(\tau, \vec{\sigma}) = \epsilon^\mu_B \Lambda^\mu{}_\nu(\tau, \vec{\sigma}) \epsilon^\nu_A, \quad U^\mu_A(\tau, \vec{\sigma}) \eta_{\mu\nu} U^\nu_B(\tau, \vec{\sigma}) = \epsilon^\mu_A \eta_{\mu\nu} \epsilon^\nu_B = \eta_{AB},$$

$$U^\mu_A(\tau, \vec{\sigma}) = \epsilon^\mu_B \Lambda^B{}_A(\tau, \vec{\sigma}) \rightarrow_{|\vec{\sigma}| \rightarrow \infty} \epsilon^\mu_A, \quad (4)$$

where $\epsilon^\mu_B = \eta_{\mu\nu} \eta^{BA} \epsilon^\nu_A$ are the inverse tetrads.

A slight generalization of these embeddings allows to find Nelson's [16] 4-coordinate transformation (but extended from $\vec{\sigma}$ -independent Lorentz transformations $\Lambda^\mu{}_\nu = \Lambda^\mu{}_\nu(\tau)$ to $\vec{\sigma}$ -dependent ones) implying Møller rotating 4-metric⁴

$$z^\mu(\tau, \vec{\sigma}) = x^\mu_o + \epsilon^\mu_A \left[\Lambda^A{}_B(\tau, \vec{\sigma}) \sigma^B + V^A(\tau, \vec{\sigma}) \right],$$

$$V^\tau(\tau, \vec{\sigma}) = \int_o^\tau d\tau_1 \Lambda^\tau{}_\tau(\tau_1, \vec{\sigma}) - \Lambda^\tau{}_\tau(\tau, \vec{\sigma}) \tau, V^r(\tau, \vec{\sigma})$$

⁴ $g_{oo} = \epsilon[(1 + \frac{\vec{a} \cdot \vec{x}}{c^2})^2 - \frac{(\vec{\omega} \times \vec{x})^2}{c^2}]$, $g_{oi} = -\epsilon \frac{1}{c} (\vec{\omega} \times \vec{x})^i$, $g_{ij} = -\epsilon \delta_{ij}$, where \vec{a} is the time-dependent acceleration of the observer's frame of reference relative to the comoving inertial frame and $\vec{\omega}$ is the time-dependent angular velocity of the observer's spatial rotation with respect to the comoving frame; \vec{x} is the position vector of a spatial point with respect to the origin of the observer's accelerated frame.

$$= \int_0^\tau d\tau_1 \Lambda^r{}_\tau(\tau_1, \vec{\sigma}) - \Lambda^r{}_\tau(\tau, \vec{\sigma}) \tau. \quad (5)$$

The Møller conditions (1) are severe restrictions on the Lorentz matrices $\Lambda(\tau, \sigma^r)$, which are stated in Ref.[9], where each Lorentz matrix Λ is represented as the product of a Lorentz boost B and a rotation matrix \mathcal{R} *to separate the translational from the rotational effects* ($\vec{\beta} = \vec{v}/c$ are the boost parameters, $\gamma(\vec{\beta}) = 1/\sqrt{1 - \vec{\beta}^2}$, $\vec{\beta}^2 = (\gamma^2 - 1)/\gamma^2$, $B^{-1}(\vec{\beta}) = B(-\vec{\beta})$; α, β, γ are three Euler angles and $R^{-1} = R^T$)

$$\Lambda(\tau, \vec{\sigma}) = B(\vec{\beta}(\tau, \vec{\sigma})) \mathcal{R}(\alpha(\tau, \vec{\sigma}), \beta(\tau, \vec{\sigma}), \gamma(\tau, \vec{\sigma})),$$

$$B^A{}_B(\vec{\beta}) = \begin{pmatrix} \gamma(\vec{\beta}) & \gamma(\vec{\beta}) \beta^s \\ \gamma(\vec{\beta}) \beta^r & \delta^{rs} + \frac{\gamma^2(\vec{\beta}) \beta^r \beta^s}{\gamma(\vec{\beta}) + 1} \end{pmatrix},$$

$$\mathcal{R}^A{}_B(\alpha, \beta, \gamma) = \begin{pmatrix} 1 & 0 \\ 0 & R^r{}_s(\alpha, \beta, \gamma) \end{pmatrix}, \quad (6)$$

where $R^r{}_s(\alpha, \beta, \gamma)$ is the standard matrix of the Euler angles.

Eqs.(1) are restrictions on the parameters $\vec{\beta}(\tau, \vec{\sigma})$, $\alpha(\tau, \vec{\sigma})$, $\beta(\tau, \vec{\sigma})$, $\gamma(\tau, \vec{\sigma})$ of the Lorentz transformations, which say that *translational accelerations and rotational frequencies are not independent but must balance each other*.

Let us consider two extreme cases.

A) *Rigid non-inertial reference frames with translational acceleration exist*. An example are the following embeddings, which are compatible with the locality hypothesis only for $f(\tau) = \tau$ (this corresponds to $\Lambda = B(\vec{0}) \mathcal{R}(0, 0, 0)$, i.e. to an inertial reference frame)

$$z^\mu(\tau, \vec{\sigma}) = x_o^\mu + \epsilon_\tau^\mu f(\tau) + \epsilon_\tau^\mu \sigma^r,$$

$$g_{\tau\tau}(\tau, \vec{\sigma}) = \epsilon \left(\frac{df(\tau)}{d\tau} \right)^2, \quad g_{\tau r}(\tau, \vec{\sigma}) = 0, \quad g_{rs}(\tau, \vec{\sigma}) = -\epsilon \delta_{rs}. \quad (7)$$

This is a foliation with parallel hyper-planes centered on a centroid $x^\mu(\tau) = x_o^\mu + \epsilon_\tau^\mu f(\tau)$ (origin of 3-coordinates). The hyper-planes have translational acceleration $\ddot{x}^\mu(\tau) = \epsilon_\tau^\mu \ddot{f}(\tau)$, so that they are not uniformly distributed like in the inertial case $f(\tau) = \tau$.

B) On the other hand *rigid rotating reference frames do not exist*. Let us consider the embedding (compatible with the locality hypothesis) with $\Lambda = B(\vec{0}) \mathcal{R}(\alpha(\tau), \beta(\tau), \gamma(\tau))$ and $x^\mu(\tau) = x_o^\mu + \epsilon_\tau^\mu \tau$, which corresponds to a foliation with parallel to space-like hyper-planes with normal $l^\mu = \epsilon_\tau^\mu$. It can be verified that it is not the inverse of an admissible 4-coordinate transformation, because

the associated $g_{\tau\tau}(\tau, \vec{\sigma})$ has a zero at $\sigma = \sigma_R$ such that ⁵

$$\sigma_R = \frac{1}{\Omega(\tau)} \left[-\dot{x}_\mu(\tau) b_r^\mu(\tau) (\hat{\sigma} \times \hat{\Omega}(\tau))^r + \sqrt{\dot{x}^2(\tau) + [\dot{x}_\mu(\tau) b_r^\mu(\tau) (\hat{\sigma} \times \hat{\Omega}(\tau))^2]^2} \right], \quad (8)$$

with $\sigma_R \rightarrow \infty$ for $\Omega \rightarrow 0$. At $\sigma = \sigma_R$ the time-like vector $z_r^\mu(\tau, \vec{\sigma})$ becomes light-like (the so-called *horizon problem* of the rotating coordinate systems), while for an admissible foliation with space-like leaves it must always remain time-like.

As shown in Ref.[9], the simplest notion of simultaneity compatible with the locality hypothesis when rotations are present is obtained with embeddings where there is a rotation matrix $R(\tau, |\vec{\sigma}|)$, namely the rotation varies as a function of some radial distance $|\vec{\sigma}|$ (*differential rotation*) from the arbitrary time-like world line $x^\mu(\tau)$, origin of the 3-coordinates on the simultaneity surfaces. Since the 3-coordinates σ^r are Lorentz scalar we shall use the radial distance $\sigma = |\vec{\sigma}| = \sqrt{\delta_{rs} \sigma^r \sigma^s}$, so that $\sigma^r = \sigma \hat{\sigma}^r$ with $\delta_{rs} \hat{\sigma}^r \hat{\sigma}^s = 1$. These embeddings are

$$z^\mu(\tau, \vec{\sigma}) = x^\mu(\tau) + \epsilon_r^\mu R^r_s(\tau, \sigma) \sigma^s \stackrel{def}{=} x^\mu(\tau) + b_r^\mu(\tau, \sigma) \sigma^r,$$

$$R^r_s(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} \delta_s^r, \quad \partial_A R^r_s(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} 0,$$

$$b_s^\mu(\tau, \sigma) = \epsilon_r^\mu R^r_s(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} \epsilon_s^\mu, \quad [b_r^\mu \eta_{\mu\nu} b_s^\nu](\tau, \sigma) = -\epsilon \delta_{rs},$$

$$R(\tau, \sigma) = R(\alpha(\tau, \sigma), \beta(\tau, \sigma), \gamma(\tau, \sigma)),$$

$$\alpha(\tau, \sigma) = F(\sigma) \tilde{\alpha}(\tau), \quad \beta(\tau, \sigma) = F(\sigma) \tilde{\beta}(\tau), \quad \gamma(\tau, \sigma) = F(\sigma) \tilde{\gamma}(\tau). \quad (9)$$

Since $z_r^\mu(\tau, \vec{\sigma}) = \epsilon_s^\mu \partial_r [R^s_u(\tau, \sigma) \sigma^u]$, it follows that the normal to the simultaneity surfaces is $l^\mu = \epsilon_r^\mu$, namely the hyper-surfaces are *parallel space-like hyper-planes*. These hyper-planes have translational acceleration $\ddot{x}^\mu(\tau)$ (it could be simulated with a rigid boost) and a rotating 3-coordinate system with rotational frequency $\Omega^r(\tau, \sigma) = -\frac{1}{2} \epsilon^{r uv} \left[R^{-1}(\tau, \sigma) \frac{\partial R(\tau, \sigma)}{\partial \tau} \right]^{uv} \rightarrow_{\sigma \rightarrow \infty} 0$.

As shown in Ref.[9], the Møller conditions (1) imply

$$0 < F(\sigma) < \frac{1}{M\sigma}, \quad \frac{dF(\sigma)}{d\sigma} \neq 0, \quad \text{or} \quad |\Omega^r(\tau, \sigma)| < \frac{m}{K\sigma} (K-1), \quad (10)$$

where the constants $m > 0$ and $K > 1$ are determined by the 4-velocity $\dot{x}^\mu(\tau)$ of the observer [$m = \min \{ \epsilon \dot{x}_\mu(\tau) l^\mu \}$, $|\vec{v}(\tau)| \leq \epsilon \dot{x}_\mu(\tau) l^\mu / K$ with $\vec{v}(\tau)$ the observer 3-velocity].

⁵We use the notations $\vec{\sigma} = \sigma \hat{\sigma}$, $\sigma = |\vec{\sigma}|$, $\vec{\Omega} = \Omega \hat{\Omega}$, $\hat{\sigma}^2 = \hat{\Omega}^2 = 1$, $\Omega^u = -\frac{1}{2} \epsilon^{urs} (\dot{R} R^{-1})^r_s$, $b_r^\mu(\tau) = \epsilon_s^\mu R^r_s(\tau)$.

Every function $F(\sigma)$ satisfying Eq.(10) gives rise to a Møller-admissible non-inertial rotating frame. As said in Subsections C and D of Section VI of Ref.[10], the choice

$$F(\sigma) = \frac{1 + \frac{\omega^2 R^2}{c^2}}{1 + \frac{\omega^2 \sigma^2}{c^2}} < \frac{2}{1 + \frac{\omega^2 \sigma^2}{c^2}} < \frac{1}{\omega \sigma}, \quad F(\sigma) \xrightarrow{c \rightarrow \infty} 1 + \frac{\omega^2 (R^2 - \sigma^2)}{c^2} + O\left(\frac{1}{c^4}\right),$$

replaces the rigid rotation $\Omega(\sigma) = \omega$ for $\sigma < R$ of a rotating disk of radius R (with $\omega R < c$) with an admissible differential rotation $\Omega(\sigma) = \omega F(\sigma)$. By varying the admissible functions $F(\sigma)$ (a gauge transformation in parametrized Minkowski theories) we can approximate the step function $\Omega(\sigma) = \omega$ for $\sigma < R$, $\Omega(\sigma) = 0$ for $\sigma > R$, as much as we wish.

In Ref.[10] there is also the treatment of the Sagnac effect in this framework.

This means that, while the linear velocities $\dot{x}^\mu(\tau)$ and the translational accelerations $\ddot{x}^\mu(\tau)$ are arbitrary, the allowed rotations $R(\alpha, \beta, \gamma)$ on the leaves of the foliation *have the rotational frequencies*, namely the angular velocities $\Omega^r(\tau, \sigma)$, *limited by an upper bound proportional to the minimum of the linear velocity* $v_l(\tau) = \epsilon \dot{x}_\mu(\tau) l^\mu$ *orthogonal to the parallel hyper-planes*.

In Refs.[9, 10] it is shown that if we consider the observers of the second skew congruence associated with these embeddings, whose unit 4-velocity is $V_\tau^\mu(\tau, \sigma^r) = z_\tau^\mu(\tau, \sigma^r) / \sqrt{\epsilon g_{\tau\tau}(\tau, \sigma^r)}$, and we endow them with an ortho-normal tetrad $V_A^\mu(\tau, \sigma^s) = (V_\tau^\mu(\tau, \sigma^s); V_r^\mu(\tau, \sigma^s))$, then we can study $dV_A^\mu(\tau, \sigma^s)/d\tau$. For each value of σ^r , namely for each observer of the congruence, we obtain an acceleration matrix and the associated *acceleration lengths*. But now, differently from the case of Fermi coordinates, the radar 4-coordinates associated with these Møller-admissible embeddings are globally defined and do not develop coordinate singularities at a distance from the observer world line of the order of the accelerations lengths.

2.3 Quantum Mechanics in Non-Inertial Frames

The postulates of non-relativistic quantum mechanics are formulated in *global inertial reference frames*, connected by the transformations of the kinematical (extended) Galilei group, which, due to the *Galilei relativity principle*, relate the observations of an inertial observer to those of another one. The self-adjoint operators on the Hilbert space, in particular the *Hamiltonian operator* (governing the time-evolution in the Schroedinger equation and identified with the *energy operator* in the projective representation of the quantum Galilei group associated with the system), correspond to the quantization of classical quantities defined in these frames. The resulting quantum theory is extremely successful *both for isolated and open systems* (viewed as sub-systems of isolated systems).

At the relativistic level conceptually nothing changes: we have the *relativity principle* stating the impossibility to distinguish special relativistic inertial frames and the kinematical Poincare' group replacing the Galilei one. Again the *energy* is one of the generators of the kinematical group and is identified with

the *canonical Hamiltonian* governing the evolution of a relativistic Schroedinger equation.

In this framework, with a semi-relativistic treatment of the electro-magnetic field we get an extremely successful theory of atomic spectra in inertial reference frames both for isolated inertial atoms (closed systems) and for accelerated ones in presence of external forces (open systems). The following cases are an elementary list of possibilities.

a) *Isolated atom* - From the time-dependent Schroedinger equation $i \frac{\partial}{\partial t} \psi = H_o \psi$, through the position $\psi = e^{i E_n t / \hbar} \psi_n$ we get the time-independent Schroedinger equation $H_o \psi_n = E_n \psi_n$ for the stationary levels and the energy spectrum E_n with its degenerations. Being isolated the atom can decay only through spontaneous emission.

b) *Atom in an external c-number, maybe time-dependent, electro-magnetic field* - Now the (energy) Hamiltonian operator is in general non conserved (open system). Only for time-independent external fields it is clear how to define the time-independent Schroedinger equation for the stationary states and the corresponding (modified) spectrum. Time-independent external electro-magnetic fields lead to removal of degeneracies (Zeeman effect) and/or shift of the levels (Stark effect). With time-dependent external fields we get the Schroedinger equation $i \frac{\partial}{\partial t} \psi = H(t) \psi$ with $H(t) = H_o + V(t)$. Therefore at each instant t the self-adjoint operator $H(t)$ defines a different basis of the Hilbert space with its spectrum, but, since in general we have $[H(t_1), H(t_2)] \neq 0$, it is not possible to define a unique associated eigenvalue equation and an associated spectrum varying continuously in t . Only when we have $[H(t_1), H(t_2)] = 0$ we can write $H(t) \psi_n(t) = E_n(t) \psi_n(t)$ with time-dependent eigenvalues $E_n(t)$ and a visualization of the spectrum as a continuous function of time. In any case, when $V(t)$ can be considered a perturbation, time-dependent perturbation theory with suitable approximations can be used to find the transition amplitudes among the levels of the unperturbed Hamiltonian H_o . Now the atom can decay both for spontaneous or stimulated emission and be excited through absorption.

c) *Atom plus an external c-number "mechanical" potential inducing, for instance, the rotational motion of an atom fixed to a rotating platform* (see the Moessbauer effect) - If the c-number potential is $V(t)$, i.e. it is only time-dependent, we have $i \frac{\partial}{\partial t} \psi = [H_o + V(t)] \psi = H(t) \psi$ with $[H(t_1), H(t_2)] = 0$ and the position $\psi = e^{i \int_o^t V(t_1) dt_1 / \hbar} \psi_1$ leads to $i \frac{\partial}{\partial t} \psi_1 = H_o \psi_1$, so that the energy levels are $E_{1n} = E_n + \int_o^t dt_1 V(t_1)$. The addition of a c-number external time-dependent electro-magnetic field leads again to the problems of case b).

d) At the relativistic level we can consider the *isolated system atom + electro-magnetic field* as an approximation to the theory of bound states in quantum electrodynamics. Both the atom and the electro-magnetic field are separately accelerated open subsystems described in an inertial frame.

In any case the modifications of the energy spectrum of the isolated atom is induced by *physical force fields* present in the inertial frame of the observer.

In case c) we can consider an accelerated observer carrying a measuring apparatus and rotating with the atom with the theory of measurement based

on the *locality hypothesis*. As a consequence the observer will detect the same spectrum as an inertial observer.

Let us consider the description of the previous cases from the point of view of a *non-inertial observer* carrying a measuring apparatus by doing a *passive coordinate transformation* adapted to the motion of the observer. Since, already at the non-relativistic level, there is *no relativity principle for non-inertial frames*, there is *no kinematical group* (larger than the Galilei group) whose transformations connect the non-inertial measurements to the inertial ones: given the non-inertial frame with its linear and rotational accelerations with respect to a standard inertial frame, we can only define the *succession* of time-dependent Galilei transformations identifying at each instant the *comoving inertial observers*, with the same measurements of the non-inertial observer if the locality hypothesis holds.

Since we are considering a *purely passive viewpoint*, there is no physical reason to expect that the atom spectra will change: there are no physical either external or internal forces but only a different viewpoint which changes the appearances and introduces the fictitious (or inertial) mass-proportional forces to describe these changes.

In the framework of parametrized Minkowski theories, like in general relativity, the passive frame-preserving diffeomorphisms on Minkowski space-time imply a special relativistic form of general covariance, but *do not form* a kinematical group (extending the Poincare' group), because there is *no relativity principle* for non-inertial observers. Therefore, there is *no kinematical generator interpretable as a non-inertial energy*.

The $c \rightarrow \infty$ limit of parametrized Minkowski theories allows to define *parametrized Galilei theories* and to describe non-relativistic congruences of non-inertial observers. Again there is no relativity principle for such observers, no kinematical group extending the Galilei one and, therefore, no kinematical generator to be identified as a non-inertial energy.

All the existing attempts [17] to extend the standard formulation of quantum mechanics from global rigid inertial frames to *special global rigid non-inertial reference frames* carried by observers with either linear (usually *constant*) acceleration or rotational (usually *constant*) angular velocity are equivalent to the definition of suitable *time-dependent unitary transformations* acting in the Hilbert space associated with inertial frames.

While in inertial frames the generator of the time evolution, namely the Hamiltonian operator H appearing in the Schroedinger equation $i \frac{\partial}{\partial t} \psi = H \psi$, also describes the energy of the system, after a time-dependent unitary transformation $U(t)$ the *generator* $\tilde{H}(t) = U(t) H U^{-1}(t) + i \dot{U}(t) U^{-1}(t)$ of the *time evolution* ⁶ in the transformed Schroedinger equation $i \frac{\partial}{\partial t} \tilde{\psi} = \tilde{H}(t) \tilde{\psi}$, with $\tilde{\psi} = U(t) \psi$, *differs from the energy operator* $H' = U(t) H U^{-1}(t)$. And also in this case like in example b), only if we have $[\tilde{H}(t_1), \tilde{H}(t_2)] = 0$ it is possible to define a unique stationary equation with time-dependent eigenvalues for $\tilde{H}(t)$.

⁶This *non-inertial Hamiltonian* containing the potential $i \dot{U} U^{-1}$ of the *fictitious or inertial forces* is not a generator of any kinematical group.

The situation is analogous to the Foldy-Wouthuysen transformation [18], which is a time-dependent unitary transformation when it exists: in this framework H' is the energy, while $\tilde{H}(t)$ is the Hamiltonian for the new Schroedinger equation and the associated S-matrix theory (theoretical treatment of semi-relativistic high-energy experiments, $\pi N, \dots$).

Since in general the self-adjoint operator $\tilde{H}(t)$ does not admit a unique associated eigenvalue equation ⁷ and, moreover, since the two self-adjoint operators \tilde{H} and H' are in general (except in the static cases) non commuting, there is *no consensus* about the results of measurements in non-inertial frames: *does a non-inertial observer see a variation of the emission spectra of atoms?* Which is the spectrum of the hydrogen atom seen by a non-inertial observer? Since for constant rotation we get $\tilde{H} = H' + \vec{\Omega} \cdot \vec{J}$, does the uniformly rotating observer see the inertial spectra or are they modified by a Zeeman effect? If an accelerated observer would actually measure the Zeeman levels with an energy measurement, this would mean that the stationary states of \tilde{H} (and not those of the inertial energy operator H') are the relevant ones. Proposals for an experimental check of this possibility are presented in Ref.[19]. Usually one says that a possible non-inertial Zeeman effect from constant rotation is either too small to be detected or masked by physical magnetic fields, so that the distinction between \tilde{H} and H' is irrelevant from the experimental point of view.

Here we have exactly the same problem like in the case of an atom interacting with a time-dependent external field: the atom is defined by its inertial spectrum, the only one unambiguously defined when $[\tilde{H}(t_1), \tilde{H}(t_2)] \neq 0$. When possible, time-dependent perturbation theory is used to find the transition amplitudes among the inertial levels. Again only in special cases (for instance time-independent \tilde{H}) a spectrum for \tilde{H} may be evaluated and usually, except in special cases like the Zeeman effect, it has no relation with the inertial spectrum (see Kuchar in Ref.[17] for an example). Moreover, also in these special cases the two operators may not commute so that the two properties described by these operators cannot in general be measured simultaneously.

As a consequence of these problems the description of measurements in non-inertial frames is often replaced by an explanation of how to correlate the phenomena to the results of measurements of the energy spectra in inertial frames. For instance in the Moessbauer effect one only considers the correction for Doppler effect (evaluated by the instantaneous comoving inertial observer) of unmodified spectra. Regarding the spectra of stars in astrophysics, only correction for gravitational red-shift of unmodified spectra are considered. After these corrections the inertial effects connected with the emission in non-inertial frames manifest themselves only in a *broadening* of the inertial spectral lines. In conclusion *atoms are always identified through their inertial spectra in absence of external fields*. The non-inertial effects, precluding the unique existence of a spectrum continuous in time, are usually small and appear as a noise over-

⁷Even when it does admit such an equation, we have $\langle \psi | H | \psi \rangle \neq \langle \tilde{\psi} | \tilde{H} | \tilde{\psi} \rangle$ and different stationary states are connected, following the treatment of the time-independent examples given by Kuchar (quoted in Refs.[17]) in which it is possible to find the spectrum of both of them, by a generalized transform.

imposed to the continuous spectrum of the center of mass.

An apparatus for measuring \tilde{H} can be an *interferometer* measuring the variation $\Delta\phi$ of the phase of the wave-function describing the two wave-packets propagating, in accordance with the non-inertial Schroedinger equation (one uses the Dirac-Feynman path integral with \tilde{H} to evaluate $\Delta\phi$) along the two arms of the interferometer. However, *the results of the interferometer only reveal the eventual non-inertial nature of the reference frame, namely they amount to a detection of the non-inertiality of the frame of reference*, as remarked in Ref.[9]. In this connection see Ref.[20] on neutron interferometry, where there is a full account of the status of the experiments for the detection of inertial effects.

Now in the non-relativistic literature there is an *active re-interpretation in terms of gravitational potentials of the previous passive view* according to a certain reading of the *non-relativistic limit of the classical (weak or strong) equivalence principle* (universality of free fall or identity of inertial and gravitational masses) and to its extrapolation to quantum mechanics (see for instance Hughes [21] for its use done by Einstein). According to this interpretation, at the classical level the *passive fictitious forces* seen by the accelerated observer are interpreted as an *active external Newtonian gravitational force* acting in an inertial frame, so that at the quantum level \tilde{H} is interpreted as the energy operator in an inertial frame in presence of an external quantum gravitational potential $\tilde{H} - H' = i\dot{U}U^{-1}$. Therefore the shift from the levels of H' to those of \tilde{H} is justified and expected. However this interpretation and use of the equivalence principle is subject to criticism already at the classical level.

A first objection is that a physical external gravitational field (without any connection with non-inertial observers) leads to the Schroedinger equation $i\frac{\partial}{\partial t}\psi = [H + V_{grav}]\psi$ and not to $i\frac{\partial}{\partial t}\tilde{\psi} = [UHU^{-1} + V_{grav}]\tilde{\psi}$, $\tilde{\psi} = U\psi$.

A further objection is that the interpretations based upon the use of the equivalence principle in the special relativistic treatment of atomic, nuclear and particle physics rely on Einstein's statements, which are explicitly referred to *static constant gravitational fields*. But according to Synge [22], static fields does not exist in general relativity: only tidal effects exist.

In Ref.[23] there is the quantization of relativistic scalar and spinning particles described by means of parametrized Minkowski theories in a class of non-inertial frames (like the ones of Eq.(9)), whose embeddings have the parametrization

$$\begin{aligned} z^\mu(\tau, \vec{\sigma}) &\approx \theta(\tau) \hat{U}^\mu(\tau) + \epsilon_a^\mu(\hat{U}(\tau)) \mathcal{A}^a(\tau, \vec{\sigma}) \\ &= x_U^\mu(\tau) + \epsilon_a^\mu(\hat{U}(\tau)) \left[\mathcal{A}^a(\tau, \vec{\sigma}) - \mathcal{A}^a(\tau, \vec{0}) \right], \\ x_U^\mu(\tau) &= z^\mu(\tau, \vec{0}) = \theta(\tau) \hat{U}^\mu(\tau) + \epsilon_a^\mu(\hat{U}(\tau)) \mathcal{A}^a(\tau, \vec{0}). \end{aligned} \quad (11)$$

The simultaneity surfaces Σ_τ are space-like hyper-planes orthogonal to the arbitrary time-like unit vector $\hat{U}^\mu(\tau)$. We see that $\theta(\tau)$ describes the freedom

in the choice of the mathematical time τ , $\partial^2 \mathcal{A}^a(\tau, \vec{0})/\partial \tau^2$ the arbitrary linear 3-acceleration of the non-inertial frame and $\frac{\partial \mathcal{A}^a(\tau, \vec{\sigma})}{\partial \tau} - \frac{\partial \mathcal{A}^a(\tau, \vec{0})}{\partial \tau}$ its angular velocity, describing the arbitrary admissible differentially rotating 3-coordinates. As a consequence, a gauge fixing for $\theta(\tau)$ and $\mathcal{A}^a(\tau, \vec{\sigma})$ realizes the choice of a well defined non-inertial frame centered on the time-like observer $x_U^\mu(\tau)$.

The positive-energy particles on Σ_τ are described by canonically conjugate 3-vectors $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$ with the particle world lines given by $x_i^\mu(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau))$ ($\epsilon p_i^2 = m_i^2$). The effective non-inertial Hamiltonian H_{ni} in one of the previous non-rigid non-inertial frames is given in Eq.(2.47) of Ref.[23]. By means of a time-dependent canonical transformation (point in the momenta) one gets (see Eq.(4.7) of Ref.[23]) the form⁸

$$H_{ni} = H_{inertial} + \sum_{i=1}^N \left[\vec{v}(\tau) + \Omega(\tau, \vec{\eta}_i(\tau)) \times \vec{\eta}_i(\tau) \right] \cdot \vec{\kappa}_i'(\tau).$$

The quantization is based on a multi-temporal quantization scheme for systems with first-class constraints, in which only the particle degrees of freedom $\eta_i^r(\tau)$, $\kappa_{ir}(\tau)$ are quantized. The gauge variables, describing the appearances (inertial effects) of the motion in non-inertial frames, are treated as c-numbers (like the time in the Schroedinger equation with a time-dependent Hamiltonian) and the physical scalar product does not depend on them. The resulting Schroedinger equation in a given non-rigid non-inertial frame of the given class is $i \hbar \frac{\partial}{\partial \tau} \psi(\tau, \vec{\eta}_i) = \hat{H}_{ni} \psi(\tau, \vec{\eta}_i)$ (see Eq.(3.71) of Ref.[23]), with the self-adjoint non-inertial Hamiltonian corresponding to a particular ordering in the quantization of the classical H_{ni} . Since the previously quoted time-dependent canonical transformation is point in the momenta it becomes a time-dependent unitary transformation at the quantum level, connecting the inertial Hamiltonian (the energy) to the non-inertial one as expected.

With this type of relativistic kinematics it has been possible to separate the center of mass⁹ and to verify that the spectra of relativistic bound states in non-inertial frames are not modified by inertial effects.

The non-relativistic limit [24] allows to recover the few existing attempts of quantization in non-inertial frames as particular cases. Now it is possible to restrict the theory to rigid non-inertial frames, where one gets for the non-inertial Hamiltonian

$$H_{ni} = \sum_i \frac{\vec{p}_i^2(t)}{2m_i} - \vec{v}(t) \cdot \vec{P}(t) - \vec{\omega}(t) \cdot \vec{J}(t)$$

(see Eq.(4.8) of Ref.[24]).

Therefore, the standard total angular momentum -rotation coupling (but not an angular momentum -linear acceleration one) emerges as the potential of inertial forces from the quantization in rigid non-inertial frames. For spinning

⁸ $\kappa_{ia}'(\tau) = \mathcal{A}_a^s(\tau, \vec{\eta}_i(\tau)) \kappa_{is}(\tau)$, where $\mathcal{A}_a^s(\tau, \sigma^u)$ is the matrix inverse of $\frac{\partial \mathcal{A}_a(\tau, \sigma^u)}{\partial \sigma^r}$.

⁹At the relativistic level this is done with a canonical transformation which is *point only* in the momenta [14, 15].

particles one gets the spin-rotation coupling, because the spin is included in the total angular momentum. However, as already said, the eigenvalues of the non-inertial Hamiltonian are a measure of non-inertiality and not of energy. At the relativistic level only non-rigid non-inertial frames are Møller admissible and this implies that the spin-rotation coupling is replaced by a much more complex potential for the relativistic inertial forces. The same complications happen also in non-relativistic non-rigid non-inertial frames.

The main open problem is the quantization of the scalar Klein-Gordon field (and of every other field) in non-inertial frames, due to the Torre and Varadarajan [25] no-go theorem, according to which in general the evolution from an initial space-like hyper-surface to a final one is *not unitary* in the Tomonaga-Schwinger formulation of quantum field theory. From the “3+1” point of view there is evolution only among the leaves of an admissible foliation and the possible way out from the theorem lies in the determination of all the admissible “3+1” splittings of Minkowski space-time satisfying the following requirements: i) existence of an instantaneous Fock space on each simultaneity surface Σ_τ (i.e. the Σ_τ ’s must admit a generalized Fourier transform); ii) unitary equivalence of the Fock spaces on Σ_{τ_1} and Σ_{τ_2} belonging to the same foliation (the associated Bogoljubov transformation must be Hilbert-Schmidt), so that the non-inertial Hamiltonian is a Hermitean operator; iii) unitary gauge equivalence of the “3+1” splittings with the Hilbert-Schmidt property. The overcoming of the no-go theorem would help also in quantum field theory in curved space-times and in condensed matter (here the non-unitarity implies non-Hermitean Hamiltonians and negative energies).

2.4 Spinning Particles in non-Inertial Frames: the Foldy-Wouthuysen Transformation and Quantization

The manifestly Lorentz covariant description of spinning particles in inertial frames at the pseudo-classical level is based on an action principle in which there are Grassmann variables ξ_i^μ , ξ_{i5} , $i=1,\dots,N$, for the description of their spin structure [26]. While we have $\xi_i^\mu \xi_i^\nu + \xi_i^\nu \xi_i^\mu = 0$, the Grassmann variables describing the spin of different particles are assumed to commute: $\xi_i^\mu \xi_j^\nu = \xi_j^\nu \xi_i^\mu$. After quantization the Grassmann variables of each particle generate the Clifford algebra of Dirac matrices: $\xi_{i5} \mapsto \sqrt{\frac{\hbar}{2}} \gamma_5$, $\xi_i^\mu \mapsto \sqrt{\frac{\hbar}{2}} \gamma_5 \gamma^\mu$. The spin tensor of each particle is $S_i^{\mu\nu} = -i \xi_i^\mu \xi_i^\nu$. Besides suitable second class constraints, there are two first class constraints associated with each particle: $\chi_{Di} = p_{i\mu} \xi_i^\mu - m_i \xi_{i5} \approx 0$, $\chi_i = p_i^2 - m_i^2 \approx 0$ with the property $\{\chi_{Di}, \chi_{Dj}\} = i \delta_{ij} \chi_i$. After quantization we have $\chi_{Di} \approx 0 \mapsto \gamma_5 (p_{i\mu} \gamma^\mu - m_i) \psi(p_i) = 0$, while the mass shell constraints become the Klein-Gordon equation implied by the square of the Dirac equation. See the second review in Refs.[11] for all the applications of pseudo-classical mechanics.

In particular in Ref.[27] there is the definition of the pseudo-classical Foldy-Wouthuysen (FW) transformation as a canonical transformation generated by a function $S_{cl} = 2i \vec{p} \cdot \vec{\xi} \xi_5 \theta(\vec{p})$ ($\text{tg } 2|\vec{p}| \theta(\vec{p}) = |\vec{p}|/m$), which after quantization

becomes the unitary FW transformation e^{iS} , $S = \beta \vec{\alpha} \cdot \vec{p} \theta(\vec{p})$ sending the Dirac Hamiltonian $H = \vec{\alpha} \cdot \vec{p} + \beta m$ into $e^{iS} H e^{-iS} = \beta \sqrt{m^2 + \vec{p}^2}$. In Ref.[27] it was possible with his technique to find an exact FW transformation for an electron interacting with a non-homogeneous magnetic field.

In Ref.[13] there was the reformulation of N charged scalar particles interacting with the electromagnetic field in the framework of parametrized Minkowski theories. Now only positive-energy scalar particles are described and the regularization of Coulomb self-energies was possible in the semiclassical approximation of using Grassmann-valued electric charges ($Q_i^2 = 0$, $Q_i Q_j = Q_j Q_i \neq 0$ for $i \neq j$). After quantization each Grassmann-valued charge gives rise to a two-level system (charge 0 and charge e or charge $-e$ and charge e). A Shanmugadhasan canonical transformation adapted to the first class constraints produced the description of the system in the radiation gauge for the electro-magnetic field, containing only transverse vector potential and electric field. In Ref.[28] the Lienard-Wiechert solution in the rest-frame instant form was found. Then in Ref.[29] it was possible to identify the semi-classical Darwin potential among the N charged scalar particles, a result that till now had been obtained only coming down from QFT through the Bethe-Salpeter equation and its instantaneous approximations.

Then in Ref.[30] there was the reformulation of N spinning particles in the framework of parametrized Minkowski theories, as the pseudo-classical basis of the positive energy $(\frac{1}{2}, 0)$ part of the $(\frac{1}{2}, \frac{1}{2})$ solutions of the Dirac equation. Essentially one has to eliminate the variables ξ_{i5} from the model of Ref.[26] and add by hand the new constraints $\phi_i \approx p_{s\mu} \xi_i^\mu \approx 0$, where the 4-momentum $p_{s\mu}$ is weakly equal to the total conserved 4-momentum of the N particle system. The global conditions $\phi_i \approx 0$ ¹⁰ turn out to belong to the set of second class constraints so that the spin structure of the positive-energy spinning particles can be described only by means of a Wigner spin-1 Grassmann 3-vectors $\vec{\xi}_i$ (the spin vector of each particle is $\vec{S}_i = -\frac{i}{2} \vec{\xi}_i \times \vec{\xi}_i$), which after quantization become the Pauli matrices: $\vec{\xi}_i \mapsto \sqrt{\frac{\hbar}{2}} \vec{\sigma}$. The resulting theory is the relativistic version of the non-relativistic Pauli equation. This is the formalism used in Ref.[23] for the quantum mechanics of spinning particles in non-inertial frames. Then in Ref.[30] there is the coupling of N charged (with Grassmann-valued charges) spinning particles to the electro-magnetic field and the determination of the Lienard-Wiechert solution on the Wigner hyper-planes of the rest-frame instant form.

Finally in Ref.[31] there was the determination of the Darwin-Salpeter potential for N positive-energy spinning particles in the scheme developed in Ref.[30]. However in Ref.[30] it was not clear whether extra coupling of the spin to the electric field were present for positive energy spinning particles. Therefore, the pseudo-classical FW transformation of Ref.[27] (in this paper the electric charge was not Grassmann-valued) was applied to positive-energy spinning particles

¹⁰All the simultaneity surface Σ_τ is involved since $p_{s\mu} = \int d^3\sigma \rho_\mu(\tau, \sigma^r)$ with $\rho_\mu(\tau, \sigma^r)$ being the momentum conjugate to the embedding $z^\mu(\tau, \sigma^r)$.

with Grassmann-valued charges interacting with an arbitrary external electromagnetic field. For a single particle the final constraint (whose quantization would produce the relativistic Pauli equation) is

$$p_o \approx Q A_o(x) + \frac{Q \vec{p} \cdot \vec{E}(x) \times \vec{S}}{(m + \sqrt{m^2 + \vec{p}^2}) \sqrt{m^2 + \vec{p}^2}} + \sqrt{m^2 + (\vec{p} - Q \vec{A}(x))^2} + 2 Q \vec{S} \cdot \vec{B}(x). \quad (12)$$

This result, valid in inertial frames, was incorporated in the formalism of Ref.[30]: after re-expressing the theory as a parametrized Minkowski theory, there was the restriction to the electro-magnetic radiation gauge and then to the rest-frame instant form on Wigner hyper-planes. The methods of Ref.[29] allowed to find the Darwin-Salpeter inter-particle potential and the correct spin-orbit and spin-spin interactions.

Most of these results have been obtained in the inertial rest-frame instant form, but, as discussed in the previous Subsection, their extension to non-inertial frames [23] can be done with time-dependent unitary transformations at the quantum level (for now in absence of the electro-magnetic field due to the Torre-Varadarajan no-go theorem for field theories on space-like hyper-surfaces). The non-rigidity of relativistic non-inertial frames replaces the spin-rotation couplings with more complicated inertial effects.

Notice that in Ref.[10], following the preliminary results of Ref.[13], there is a study of Maxwell theory as a parametrized Minkowski theory. There are indications that in uniformly rotating (Møller forbidden) non-inertial frames the non-inertial electric and magnetic fields (used for the magnetosphere of pulsars [32, 33]) are not connected to the inertial ones by Lorentz transformations. A better understanding and the connection with Mashhoon's nonlocal electrodynamics [34] is needed.

2.5 The Multipolar Expansion and the Mathisson-Papapetrou-Dixon-Souriau Equations

In Refs.[29, 31] there was the evaluation of the energy-momentum tensor of the isolated system of N positive-energy either scalar or spinning particles plus the electro-magnetic field in the radiation gauge on the Wigner hyper-planes of the inertial rest-frame instant form. Then the description of Dixon multipolar expansion in the rest-frame instant form done in the third paper of Ref.[14] allows to study the Mathisson-Papapetrou-Dixon-Souriau pole-dipole equations (see the next Section) in a consistent Hamiltonian framework both for isolated systems and for their open subsystems. For open subsystems the subsidiary conditions needed by these equations are automatically selected by the Hamiltonian formalism, once a choice is made for the collective variable to be used as a centroid in the pole-dipole approximation of the extended open subsystem. The drawback of these multipolar expansions is that any set of variables one chooses to describe the centroid (monopole), the spin (dipole) and the higher

multi-poles do not form a canonical basis of the phase space: their complicated Poisson brackets reduce the utility of the multipolar approximation. Moreover the Hamiltonian description of extended systems is strictly speaking equivalent to the multipolar expansion only when the energy-momentum is an analytic function of its variables [35].

2.6 The Dirac Field in Non-Inertial Frames

In Ref.[36] there is the reformulation of Dirac fields as parametrized Minkowski theories with a special emphasis on the study of the highly non-trivial algebra of first and second class constraints ¹¹ on arbitrary admissible simultaneity space-like 3-surfaces. A Grassmann valued Dirac spinor was used, so that after quantization one can recover the anti-commuting Dirac fields of QFT. However the same algebra of constraints is obtained for the Dirac equation in first quantization. The starting point was a special form of the Lagrangian, which is possible only in the flat Minkowski space-time where the 4-spin connection vanishes (see Appendix A of Ref.[36]). After the analysis of the constraints in non-inertial frames, the main results are stated in the inertial rest-frame instant form. However the determination of the non-inertial Hamiltonian in a fixed admissible “3+1” splitting should be obtainable with a time-dependent unitary transformation following the scheme of Ref.[23]. This would be the “3+1” point of view on the Dirac equation in non-inertial frames to be compared with the results (valid locally around the accelerated observer) of the 1+3 point of view quoted in Refs.[3] (see in particular the results of Hehl and Ni [37]). Again the spin-rotation couplings should become complicated inertial effects depending on the chosen non-rigid non-inertial frame.

3 Spin-Rotation Couplings in General Relativity

3.1 The 1+3 Splitting of the Space-Time

In general relativity a reference frame is defined by a congruence of time-like curves, say the set of the world lines of a family of observers. We denote by u the unit tangent vector ($u \cdot u = -1$) of the world lines of a generic reference frame. The splitting of the space-time along u and its orthogonal local rest space LRS_u gives the measurement relative to u of any tensor field defined on a domain of the space-time; similarly one can obtain the formulation relative to u of any tensor equation ¹².

¹¹Second class constraints appear because the Dirac equation is a first order partial differential equation.

¹²In this and the next sections units are chosen here in order that the speed of light in empty space satisfies $c = 1$; moreover the metric signature conventions is fixed assuming $\epsilon = -1$ whereas, for notations and conventions, we follow [38].

The measurement of the space-time metric gives rise to the spatial metric $P(u)_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$, i.e., the spatial projection operator with respect to u ; the temporal projection operator along u is $-u \otimes u$. Analogously $\eta(u)_{\alpha\beta\sigma} = u^\rho \eta_{\rho\alpha\beta\sigma}$ is the only spatial field resulting from the measurement of the unit (oriented) volume 4-form η ; it defines the spatial cross product \times_u as well as the spatial dual operation on LRS_u .

From the measurement of a p -form S only two distinct fields result: the purely spatial or “magnetic” part $S(u)^{(M)}$ (a p -form) and the “electric” part $S(u)^{(E)}$ (a $(p-1)$ -form), the spatial projection of any contraction with a single factor of u :

$$S = u^\flat \wedge S(u)^{(E)} + S(u)^{(M)} , \quad (13)$$

where the completely covariant (contravariant) form of a generic tensor X is denoted by X^\flat (X^\sharp).

For a generic 2-form S the component notation for this decomposition is

$$S_{\alpha\beta} = 2u_{[\alpha} S(u)^{(E)}_{\beta]} + S(u)^{(M)}_{\alpha\beta} = 2u_{[\alpha} S(u)^{(E)}_{\beta]} + \eta(u)_{\alpha\beta}{}^\sigma S(u)^{(M)}_\sigma , \quad (14)$$

where $S(u)^{(M)}_\sigma$ denotes the spatial dual of $S(u)^{(M)}_{\alpha\beta}$.

According to the measurement process, the spatial and temporal projection of space-time derivative operators gives rise to the corresponding spatial and temporal counterparts. As described in [38] the spatial covariant derivative is defined as $\nabla_{(u)\alpha} = P(u)P(u)^\beta_\alpha \nabla_\beta$ (the first projection operator acts on the tensorial indices of the field after the derivative is applied) and the spatial Lie derivative along a generic field X is $\mathcal{L}_{(u)}X = P(u)\mathcal{L}_X$. Similarly one can construct temporal derivatives: the Lie temporal derivative $\nabla_{(\text{lie}, u)} = P(u)\mathcal{L}_u$, the Fermi-Walker temporal derivative $\nabla_{(\text{fw}, u)} = P(u)\nabla_u$.

The covariant derivative of u has the following decomposition:

$$\nabla_\alpha u^\beta = \eta(u)_\alpha{}^\beta{}_\mu \omega(u)^\mu + \theta(u)_\alpha{}^\beta - u_\alpha a(u)^\beta , \quad (15)$$

where $a(u) = \nabla_{(\text{fw}, u)} u \in LRS_u$ is the acceleration vector, $\theta(u) \in LRS_u \otimes LRS_u$ is the (symmetric) deformation 2-tensor and $\omega(u) \in LRS_u$ is the vorticity vector of the observer congruence.

If X is a tensor field defined only along the line ℓ_U , (parametrized by the proper time τ_U and having unit tangent U), then the “measurement” of $\frac{DX}{d\tau_U}$, intrinsic derivative along U , by a family of observers with 4-velocity u is:

$$\frac{D_{(\text{fw}, U, u)}}{d\tau_{(U, u)}} X = [\nabla_{(\text{fw}, u)} + \nabla_{(u)\nu(U, u)}] X , \quad (16)$$

where $U = \gamma(U, u)[u + \nu(U, u)]$, $d\tau_{(U, u)} = \gamma(U, u)d\tau_U$ is the differential of the standard relative time parametrization along the line U , and the field X on the right hand side is some smooth extension of X to a neighborhood of the line.

It is convenient to introduce the composition of projection maps from the local rest space of an observer onto that of another:

$$P(U, u) = P(U)P(u) : LRS_u \longrightarrow LRS_U .$$

We also introduce the relative gravitational field $F_{(\text{fw}, U, u)}^{(G)}$ (see [38]):

$$\begin{aligned} F_{(\text{fw}, U, u)}^{(G)} &= -\gamma(U, u)^{-1} P(u) \frac{Du}{d\tau_U} = -P(u) \frac{Du}{d\tau_{(U, u)}} \\ &= \gamma(U, u) \left[g(u) + \|\nu(U, u)\| \left(\frac{1}{2} \hat{\nu}(U, u) \times_u H(u) + \right. \right. \\ &\quad \left. \left. - \theta(u) \mathbf{L} \hat{\nu}(U, u) \right) \right], \end{aligned} \quad (17)$$

where $g(u) = -a(u)$ represents the gravito-electric field while $H(u) = 2\omega(u)$ is the gravito-magnetic field; \mathbf{L} denotes right-contraction.

The last equation of (17) gives the gravitational force a Lorentz-like form allowing the introduction of terms such as gravito-electromagnetic force and gravito-electromagnetism.

The Riemann tensor is represented by its “electric” ($\mathcal{E}(u)_{\alpha\beta}$), “magnetic” ($\mathcal{H}(u)_{\alpha\beta}$) and “mixed” ($\mathcal{F}(u)_{\alpha\beta}$) parts, namely

$$\begin{aligned} \mathcal{E}(u)_{\beta\delta} &= R_{\alpha\beta\gamma\delta} u^\alpha u^\gamma, \\ \mathcal{H}(u)_{\beta\delta} &= -\frac{1}{2} \eta(u)^{\gamma\mu}{}_\delta R_{\alpha\beta\gamma\mu} u^\alpha, \\ \mathcal{F}(u)_{\beta\delta} &= \frac{1}{4} \eta(u)^{\alpha\mu}{}_\beta \eta(u)^{\gamma\sigma}{}_\delta R_{\alpha\mu\gamma\sigma}. \end{aligned} \quad (18)$$

This summarizes the essential “1+3” splitting formalism. It will be used in the next section.

3.2 Spinning Particles in External Gravitational Fields

3.2.1 Extended Bodies in Classical General Relativity

In general relativity, an extended body is described by its associated energy momentum tensor. A small body can be studied by a multi-pole expansion method: the body is equivalently described by a set of multi-pole (energy) moments defined along a central line [39, 40, 41, 42, 43]. The cutoff at successive multi-pole orders defines a hierarchy of elementary multi-pole particles (see e.g. [39, 40, 44, 45]). The first step is the point particle (or single-pole), governed by the geodesic equation of motion. The second one is the dipole (“spinning”) particle which interests us here. The equations of motion for such a particle were first derived in the pure gravitational case by Papapetrou [40]:

$$\begin{aligned} \frac{D}{d\tau_U} p^\alpha &= -\frac{1}{2} R^\alpha{}_{\beta\rho\sigma} U^\beta S^{\rho\sigma} \\ \frac{D}{d\tau_U} S^{\alpha\beta} &= p^\alpha U^\beta - p^\beta U^\alpha; \end{aligned} \quad (19)$$

where $R_{\alpha\beta\rho\sigma}$ is the Riemann tensor, p^α is the (generalized) momentum vector, $S^{\alpha\beta}$ is a (antisymmetric) spin tensor, $U = DX/d\tau_U$ is the unit tangent vector

($U^\alpha U_\alpha = -1$) of the “center line” ℓ_U used to make the multi-pole reduction, and where $X = X(\tau_U)$ is the center point whose world line is ℓ_U . The fields S , U and p are defined only along ℓ_U .

The case in which both gravitational and electromagnetic fields are present was studied by Dixon and Souriau [44, 46].

It is well known that the number of independent equations in (19) is less than that of the unknown quantities: 3 additional scalar supplementary conditions (SC) are needed for the scheme to be completed. Once a suitable choice has been made, ℓ_U , p and S can, in principle, be determined by the equations. The various supplementary conditions which are considered in the literature are all of the form $\hat{u}^\alpha S_{\alpha\beta} = 0$ for some time-like unit vector \hat{u} along the world line ℓ_U . According to the special relativistic analogy ([47], p. 161), this is equivalent to defining the central line ℓ_U as the world line of the centroid of the body with respect to an observer family with 4-velocity \hat{u} .

The supplementary conditions most frequently discussed in the literature are

(CP) $\hat{u} = u$ (Corinaldesi-Papapetrou condition: see e.g. [48, 49]), where u is a (known) preferred family of observers usually suggested by the background;

(T) $\hat{u} = p/||p|| = \bar{u}$ (Tulczyjew’s condition: see e.g. [50, 41, 44, 51]);

(P) $\hat{u} = U$ (Pirani’s condition: see e.g. [52, 53]).

Clearly the fields ℓ_U , p and S all depend on the choice of supplementary conditions [51] so a more precise notation would be: $X_{(SC)}$, $U_{(SC)}$, $p_{(SC)}$, $S_{(SC)}$, where the index values $SC = CP, T, P$ correspond to these choices. This cumbersome notation (but not confusion) is usually avoided.

It is clear from the above discussion that the most widely accepted description of spinning test particles in relativity is far from being complete, at least for the arbitrary choice of supplementary conditions required to make the model compatible.

Fortunately, the Hamiltonian methods of the previous Section, when suitably extended to include gravity, will allow the general relativistic canonical formulation of these results. In the first of Ref. [14] there is a detailed special relativistic study of the possible effective center of motion of an open subsystem of an isolated system. It turns out that at the hamiltonian level the most convenient choices are either the Møller energy center of motion or the Tulczyjew’s one.

3.2.2 The Mathisson-Papapetrou-Dixon-Souriau Equations of Motion

The Papapetrou-Dixon-Souriau equations of motion of a spinning test particle in a given gravitational and electromagnetic background [54, 55, 56, 46] are given by:

$$\begin{aligned}
\frac{D}{d\tau_U} p^\alpha &= -\frac{1}{2} R^\alpha_{\beta\rho\sigma} S^{\rho\sigma} U^\beta + e F^\alpha_{\beta} U^\beta - \frac{1}{2} \lambda S^{\mu\nu} \nabla^\alpha F_{\mu\nu} , \\
\frac{D}{d\tau_U} S^{\alpha\beta} &= p^\alpha U^\beta - p^\beta U^\alpha + \lambda [S^{\alpha\mu} F_\mu{}^\beta - S^{\beta\mu} F_\mu{}^\alpha] ,
\end{aligned} \tag{20}$$

where $F^{\alpha\beta}$ is the electromagnetic field and λ is an electromagnetic coupling scalar.

The spatial dual of the spin-electromagnetism coupling term $S^{\alpha\mu} F_\mu{}^\beta - S^{\beta\mu} F_\mu{}^\alpha$ appearing in the second of equations (20) coincides with the coupling term found by Bargman, Michel and Telegdi [57]. The classic Papapetrou's scheme is obtained from (20) by assuming $F = 0$, i.e. by neglecting the electromagnetic field.

The various terms arising from the coupling with the spin of the gravitational, electromagnetic and inertial fields can be derived after a systematic use of splitting techniques.

Precisely, it is convenient to introduce the following notation for the spin-gravity and spin-electromagnetism coupling terms:

$$\begin{aligned}
\mathcal{R}_{\alpha\beta} &= \frac{1}{2} R_{\alpha\beta\mu\nu} S^{\mu\nu} , \\
Q_\alpha &= \frac{1}{2} S^{\mu\nu} \nabla_\alpha F_{\mu\nu} , \\
N^{\alpha\beta} &= S^{\alpha\mu} F_\mu{}^\beta - S^{\beta\mu} F_\mu{}^\alpha .
\end{aligned} \tag{21}$$

Assume a) $p = ||p||\bar{u} = M_0\bar{u}$, defined along ℓ_U , is time-like: $\bar{u}^\alpha\bar{u}_\alpha = -1$ and b) U and \bar{u} may be extended in a regular way in a neighborhood of the line ℓ_U . We then have two time-like congruences, \bar{u} and U , at our disposal.

Both \bar{u} and U are associated with the particle in a natural way and so are both candidates for defining the proper rest frame of the particle. There is no agreement in the literature about which of them should define the proper rest frame. This also leads to the problem of definition of the center of mass world line as the P-center or the T-center respectively.

Let us now consider a generic observer field u . The splitting of U and p along u and $LR S_u$ is:

$$\begin{aligned}
U &= \gamma(U, u)[u + \nu(U, u)], \\
p &= ||p||\bar{u} = E(p, u)u + p(u) = M_0\gamma(\bar{u}, u)[u + \nu(\bar{u}, u)],
\end{aligned} \tag{22}$$

where $E(p, u) = \gamma(\bar{u}, u)||p|| = \gamma(\bar{u}, u)M_0$. Let us introduce the following notation for the electric and magnetic parts of the antisymmetric 2-tensor fields S , F , N and \mathcal{R} which appear in the equations: $L(u) = S(u)^{(E)}$, $S(u) = S(u)^{(M)}$, $E(u) = F(u)^{(E)}$, $B(u) = F(u)^{(M)}$, $N(u) = N(u)^{(E)}$, $M(u) = N(u)^{(M)}$, $K(u) = \mathcal{R}(u)^{(E)}$, $T(u) = \mathcal{R}(u)^{(M)}$, and extending the notation to the 1-form Q , $Q(u) = Q(u)^{(M)}$, $q(u) = Q(u)^{(E)}$: $Q = Q(u) + q(u)u$, $q(u) = -u \cdot Q$. In particular, we have:

$$K(u)^\alpha = \mathcal{E}(u)^\alpha_\sigma L(u)^\sigma + \mathcal{H}(u)^\alpha_\sigma S(u)^\sigma ,$$

$$\begin{aligned}
T(u)^\alpha &= \mathcal{F}(u)_\mu^\alpha S(u)^\mu - \mathcal{H}(u)_\mu^\alpha L(u)^\mu , \\
N(u)^\alpha &= [E(u) \times_u S(u) + B(u) \times_u L(u)]^\alpha , \\
M(u)^\alpha &= [B(u) \times_u S(u) + L(u) \times_u E(u)]^\alpha , \\
Q(u)^\alpha &= S(u)^\sigma \nabla_{(u)}^\alpha B(u)_\sigma - L(u)^\sigma \nabla_{(u)}^\alpha E(u)_\sigma \\
&\quad + [N(u) \times_u \omega(u)]^\alpha + \theta(u)^\alpha_\mu N(u)^\mu , \\
q(u) &= L(u)^\sigma \nabla_{(\text{fw}, u)} E(u)_\sigma - S(u)^\sigma \nabla_{(\text{fw}, u)} B(u)_\sigma - N(u) \cdot a(u) . \quad (23)
\end{aligned}$$

The background is assumed to be completely known; in particular the electromagnetic field satisfies the Maxwell equations. It is also useful to write the following decomposition for $K(U) \in LRS_U$:

$$K(U) = P(u, U)K(U) + u[\nu(U, u) \cdot P(U, u)K(U)] .$$

We have $P(u, U)K(U) = \gamma(U, u)[K(u) + \nu(U, u) \times_u T(u)]$.

With these definitions, the equations of motion (20) are equivalent to the following set:

$$\begin{aligned}
\gamma(U, u) \frac{D_{(\text{fw}, U, u)}}{d\tau_{(U, u)}} [E(p, u)\nu(\bar{u}, u)] &= E(p, u)F_{(\text{fw}, U, u)}^{(G)} + P(u, U)[-K(U) + eE(U)] \\
&\quad - \lambda Q(u) , \quad (24)
\end{aligned}$$

$$\begin{aligned}
\gamma(U, u) \frac{D_{(\text{fw}, U, u)}}{d\tau_{(U, u)}} E(p, u) &= E(p, u)\nu(\bar{u}, u) \cdot F_{(\text{fw}, U, u)}^{(G)} \\
&\quad + \nu(U, u) \cdot P(U, u)[-K(U) + eE(U)] - \lambda q(u) , \quad (25)
\end{aligned}$$

$$\begin{aligned}
\gamma(U, u) \frac{D_{(\text{fw}, U, u)}}{d\tau_{(U, u)}} L(u) &= S(u) \times_u F_{(\text{fw}, U, u)}^{(G)} + \gamma(U, u)E(p, u)[\nu(U, u) - \nu(\bar{u}, u)] \\
&\quad + \lambda N(u) , \quad (26)
\end{aligned}$$

$$\begin{aligned}
\gamma(U, u) \frac{D_{(\text{fw}, U, u)}}{d\tau_{(U, u)}} S(u) &= F_{(\text{fw}, U, u)}^{(G)} \times_u L(u) + \gamma(U, u)E(p, u)\nu(\bar{u}, u) \times_u \nu(U, u) \\
&\quad + \lambda M(u) . \quad (27)
\end{aligned}$$

The reference frame field u appearing in eqs (24)–(27) is a generic one. If one specializes it to U , following simplifications occur:

$$\begin{aligned}
D_{(\text{fw}, U, U)}/d\tau_{(U, U)} &= \nabla_{(\text{fw}, U)} , \\
\nu(U, U) &= 0 , \quad \gamma(U, U) = 1 , \\
F_{(\text{fw}, U, U)}^{(G)} &= -a(U) ,
\end{aligned}$$

so that we have

$$\begin{aligned}
\nabla_{(\text{fw}, U)} [E(p, U)\nu(\bar{u}, U)] &= -E(p, U)a(U) - K(U) + eE(U) - \lambda Q(U) , \\
\nabla_{(\text{fw}, U)} E(p, U) &= -E(p, U)\nu(\bar{u}, U) \cdot a(U) - \lambda q(U) , \\
\nabla_{(\text{fw}, U)} L(U) &= -S(U) \times_U a(U) - E(p, U)\nu(\bar{u}, U) + \lambda N(U) , \\
\nabla_{(\text{fw}, U)} S(U) &= -a(U) \times_U L(U) + \lambda M(U) . \quad (28)
\end{aligned}$$

If we instead specialize the reference frame field to \bar{u} , one has $\nu(\bar{u}, \bar{u}) = 0$ so that

$$M_0 F_{(\text{fw}, U, \bar{u})}^{(G)} = -P(\bar{u}, U)[-K(U) + eE(U)] + \lambda Q(\bar{u}) , \quad (29)$$

$$\gamma(U, \bar{u}) D_{(\text{fw}, U, \bar{u})} / d\tau_{(U, \bar{u})} M_0 = \nu(U, \bar{u}) \cdot P(\bar{u}, U)[-K(U) + eE(U)] - \lambda q(\bar{u}) , \quad (30)$$

$$\begin{aligned} \gamma(U, \bar{u}) D_{(\text{fw}, U, \bar{u})} / d\tau_{(U, \bar{u})} L(\bar{u}) &= S(\bar{u}) \times_{\bar{u}} F_{(\text{fw}, U, \bar{u})}^{(G)} + \gamma(U, \bar{u}) E(p, \bar{u}) \nu(U, \bar{u}) \\ &\quad + \lambda N(\bar{u}) , \end{aligned} \quad (31)$$

$$\gamma(U, \bar{u}) D_{(\text{fw}, U, \bar{u})} / d\tau_{(U, \bar{u})} S(\bar{u}) = F_{(\text{fw}, U, \bar{u})}^{(G)} \times_{\bar{u}} L(\bar{u}) + \lambda M(\bar{u}) . \quad (32)$$

Taking into account equation (29), one can rewrite equation (30) as an energy theorem:

$$\gamma(U, \bar{u}) D_{(\text{fw}, U, \bar{u})} / d\tau_{(U, \bar{u})} M_0 = -M_0 \nu(U, \bar{u}) \cdot F_{(\text{fw}, U, \bar{u})}^{(G)} - \lambda [Q(\bar{u}) \cdot \nu(U, \bar{u}) + q(\bar{u})] . \quad (33)$$

The motion of the body relative to u can in principle be described by either the spatial velocity $\nu(U, u)$ or the generalized one $\nu(\bar{u}, u)$, the two descriptions being inequivalent. From ordinary relativistic kinematics we have the following “addition of velocity law”

$$\frac{\gamma(U, \bar{u})}{\gamma(U, u)} \nu(U, \bar{u}) = P(\bar{u}, u) [\nu(U, u) - \nu(\bar{u}, u)] . \quad (34)$$

Analogously, the acceleration of the particle relative to u can also be described by the Fermi-Walker spatial derivative of either $\nu(U, u)$ or $\nu(\bar{u}, u)$, and again the two are inequivalent. The generalized acceleration corresponding to $\nu(\bar{u}, u)$ is given by equation (24). On the right hand side of (24) are terms corresponding to the gravitational field, the electromagnetic field, and a sum of spin-gravity- electromagnetism coupling terms. It is convenient to introduce the following (relative) tidal acceleration, due entirely to the spin of the particle, to represent this coupling:

$$\begin{aligned} \gamma(U, u) A(\bar{u}, u) &= -P(u, U) K(U) - \lambda Q(u) \\ &= \gamma(U, u) D_{(\text{fw}, U, u)} / d\tau_{(U, u)} [E(p, u) \nu(\bar{u}, u)] \\ &\quad - E(p, u) F_{(\text{fw}, U, u)}^{(G)} - e P(u, U) E(U) . \end{aligned} \quad (35)$$

The choice of the supplementary condition is fundamental: 1) it defines the world line ℓ_U , support for all the fields we are dealing with; 2) the equations are not formally invariant for different choices. In fact, from (22) in the P,T,CP cases we have respectively

$$(P): \quad L(u) = S(u) \times_u \nu(U, u) , \quad (36)$$

$$(T): \quad L(u) = S(u) \times_u \nu(\bar{u}, u) , \quad (37)$$

$$(CP): \quad L(u) = 0 . \quad (38)$$

When substituted into (24) these three conditions lead to a total of 6 different expressions for the sum of the spin-gravity and the spin-electromagnetism couplings.

3.2.3 Pseudo-Classical Mechanics

In the “3+1” framework of pseudo-classical mechanics [26], instead, a study was made of the coupling of a spinning particle to an external gravitational field in Ref.[58]. It was shown that the algebra of constraints closes consistently only if the external gravitational field is *torsionless*. The Pauli-Lubanski spin 4-vector Σ^μ of the spinning particle satisfies the equation of motion $\frac{d\Sigma^\mu(\tau)}{d\tau} + \Gamma_{\lambda\nu}^\mu \dot{x}^\nu \Sigma^\lambda = 2 g^{\mu\rho} R_{\rho\nu} \dot{x}^\nu \Gamma_5$ ($\Gamma_5 = \xi_0 \xi_1 \xi_2 \xi_3 \mapsto \gamma_5$): the spin vector is not parallel transported where the Ricci tensor is non-zero. This is an analogue of the Bargmann-Michel-Telegdi equation in external electro-magnetic fields [57]. After quantization the Dirac equation has the standard form $(i \hbar \gamma^\mu \nabla_\mu - m) \psi(x) = 0$.

3.3 The Rest-Frame Instant Form of metric and tetrad Gravity in the 3+1 Point of View.

In Ref.[12], after a study of the constraints of ADM canonical metric gravity, the same was done for ADM tetrad gravity, the theory needed for the coupling of fermions to gravity.

The Hamiltonian formulation of both metric and tetrad gravity is well defined in globally hyperbolic space-times which are asymptotically flat with suitable boundary conditions at spatial infinity so that the only existing asymptotic symmetries are associated with the ADM Poincaré generators. There are no Killing vectors and the Dirac Hamiltonian turns out to be the weak ADM energy.

3.3.1 The Foldy-Wouthuysen Transformation for Spinning Particles in an External Tetrad Field.

In Ref.[58] there was no study of a pseudo-classical FW transformation along the lines of Ref. [27]. Therefore, following Ref.[12], we outline some original results of a novel study involving a spinning particle coupled to an external cotetrad field [59]. We start from the Lagrangian of Ref.[58] coupled to an external tetrad field [(A) are flat indices]

$$\begin{aligned}
L(\tau) = & -\frac{i}{2} \xi_5 \dot{\xi}_5 - \frac{i}{2} \eta_{(A)(B)} \xi^{(A)} \left(\dot{\xi}^{(B)} + \omega_{\mu(C)}^{(B)} \dot{x}^\mu \xi^{(C)} \right) - \\
& - mc \sqrt{\eta_{(A)(B)} \left(E_\mu^{(A)}(x) \dot{x}^\mu - \frac{i}{mc} \xi^{(A)} \dot{\xi}_5 \right) \left(E_\nu^{(B)}(x) \dot{x}^\nu - \frac{i}{mc} \xi^{(B)} \dot{\xi}_5 \right)}.
\end{aligned} \tag{39}$$

Here $E_\mu^{(A)}(x)$ and $E_{(A)}^\mu(x)$ are cotetrad and tetrad fields, respectively, evaluated at the particle position and satisfying

$$E_\mu^{(A)} E_\nu^{(A)} = \delta_\nu^\mu, \quad E_\mu^{(A)} E_{(B)}^\mu = \delta_{(B)}^{(A)}, \quad E_{(A)}^\mu {}^4 g_{\mu\nu} E_{(B)}^\nu = \eta_{(a)(b)}.$$

They describe an external gravitational field, whose 4-metric is ${}^4 g_{\mu\nu}(x) = E_\mu^{(A)}(x) \eta_{(A)(B)} E_\nu^{(B)}(x)$, as a theory of dynamical time-like observers endowed with spatial triads. The flat limit corresponds to $E_\mu^{(A)} \rightarrow \delta_\mu^{(A)}$.

The spin connection is

$$\omega_{\mu(B)}^{(A)} = E_\alpha^{(A)} \nabla_\mu E_{(B)}^\alpha = E_\alpha^{(A)} \left(\partial_\mu E_{(B)}^\alpha + {}^4 \Gamma_{\mu\nu}^\alpha E_{(B)}^\nu \right). \quad (40)$$

In phase space, after the elimination of the second class constraints, we remain with the variables $x^\mu, p_\mu, \xi^{(A)}, \xi_5$ satisfying the Dirac brackets $\{x^\mu, p_\nu\}^* = -\delta_\nu^\mu$, $\{\xi^{(A)}, \xi^{(B)}\}^* = i \eta^{(A)(B)}$, $\{\xi_5, \xi_5\}^* = -i$. There are the following two first class constraints [$\omega_{(A)(B)(C)}$ are the Ricci rotation coefficients]

$$\begin{aligned} \chi &= {}^4 g^{\mu\nu} P_\mu P_\nu - m^2 c^2 = \eta^{(A)(B)} P_{(A)} P_{(B)} - m^2 c^2 \approx 0, \\ \chi_D &= P_\mu E_{(A)}^\mu \xi^{(A)} - mc \xi_5 = P_{(A)} \xi^{(A)} - mc \xi_5 \approx 0, \end{aligned}$$

$$\begin{aligned} P_\mu &= p_\mu - \frac{i}{2} \omega_{\mu(A)(B)}(x) \xi^{(A)} \xi^{(B)}, \\ P_{(A)} &= E_{(A)}^\mu(x) P_\mu = p_{(A)} - \frac{i}{2} \omega_{(A)(B)(C)}(x) \xi^{(B)} \xi^{(C)}. \end{aligned} \quad (41)$$

They satisfy the algebra $\{\chi_D, \chi_D\}^* = i\chi$, $\{\chi, \chi_D\}^* = \{\chi, \chi\}^* = 0$.

The Dirac Hamiltonian is $H_D = \lambda(\tau) \chi + \lambda_\xi(\tau) \chi_D$, where $\lambda(\tau)$ and $\lambda_\xi(\tau)$ are Dirac multipliers, even and odd respectively. As shown in Ref.[27], in Minkowski space-time in the gauge $x^o \approx \tau$ the Dirac Hamiltonian becomes $H_D = p_o + \tilde{\lambda} \xi^{(o)} \chi_D$ with a constant $\tilde{\lambda}$: if we choose $\tilde{\lambda} = 2/\hbar$ the quantization of the Grassmann variables produces the Hamiltonian $H = \vec{\alpha} \cdot \vec{p} + \beta m$ associated with the free Dirac equation. The FW transformation transforms it to $H' = \beta \sqrt{m^2 c^2 + \vec{p}^2}$.

Since in the free case we have $\chi_D \mapsto \gamma_5 (p_\mu \gamma^\mu - mc)$, the FW transformation sends $\chi_D = p_\mu \xi^\mu - mc \xi_5$ into $\chi_D' = p_o \xi^o - mc \xi_5$, which gives rise to the quantum Hamiltonian $\gamma_5 H' = \gamma_5 \beta \sqrt{m^2 c^2 + \vec{p}^2}$.

These results are based on the fact that after quantization in the Dirac-Pauli representation for the Dirac matrices $\xi^{(o)}$ and ξ_5 become anti-diagonal matrices, while $\xi^{(a)}$ and $\xi^{(o)} \xi_5$ become diagonal.

Therefore, the Dirac constraint can be written in the form

$$\chi_D = P_{(A)} \xi^{(A)} - mc \xi_5 = p_{(o)} \xi^{(o)} - mc \xi_5 - \mathcal{E}_5(\xi) - \mathcal{O}_5(\xi),$$

$$\begin{aligned}
\mathcal{E}_5(\xi) &= -\frac{i}{2} \omega_{(o)(b)(c)} \xi^{(b)} \xi^{(c)} \xi^{(o)} - i \omega_{(a)(o)(b)} \xi^{(o)} \xi^{(b)} \xi^{(a)}, \\
\mathcal{O}_5(\xi) &= -\tilde{P}_{(a)} \xi^{(a)} = -[p_{(a)} - \frac{i}{2} \omega_{(a)(b)(c)} \xi^{(b)} \xi^{(c)}] \xi^{(a)}, \tag{42}
\end{aligned}$$

with the even $\mathcal{E}_5(\xi)$ and odd $\mathcal{O}_5(\xi)$ parts going to diagonal and anti-diagonal matrices respectively in the final quantum Hamiltonian H' .

We look for a FW transformation such that the new odd part $\mathcal{O}'_5(\xi)$ is of the order $O(\frac{1}{mc})$. Then one could devise an iterative scheme of FW transformations such that at the $n - th$ step the odd part becomes of order $O(\frac{1}{(mc)^n})$.

Preliminary calculations [59] for the pseudo-classical FW canonical transformation $\chi_D \mapsto \chi'_D = e^{\{\cdot, s\}} \chi_D$ have been done. It turns out that the generating function S has the form $[S^{(a)} = \frac{i}{2} \epsilon^{(a)}_{(b)(c)} \xi^{(b)} \xi^{(c)}$ is the spin vector; we work in the Schwinger time-gauge with tetrads adapted to the “3+1” splitting of space-time]

$$\begin{aligned}
S &= -2i P_{(a)} \xi^{(a)} \xi_5 \theta(\alpha), \\
\theta(\alpha) &= \frac{1}{2\sqrt{\alpha}} \arctg \frac{\sqrt{\alpha}}{mc}, \\
\alpha &= i \{P_{(a)} \xi^{(a)}, P_{(b)} \xi^{(b)}\}^* = \\
&= -\eta^{(a)(b)} p_{(a)} p_{(b)} + p_{(a)} \eta^{(a)(b)} \omega_{(b)(c)(d)} \epsilon^{(c)(d)}_{(e)} S^{(e)} + \\
&+ 2p_{(o)} \omega_{(a)(b)(o)} \epsilon^{(a)(b)}_{(c)} S^{(c)} + 2i \xi^{(o)} \xi^{(a)} \left[\omega_{(o)(a)(o)} \omega_{(b)(c)(o)} \epsilon^{(b)(c)}_{(d)} S^{(d)} + \right. \\
&+ \left. \left(p_{(b)} - \frac{1}{2} \omega_{(b)(c)(d)} \epsilon^{(c)(d)}_{(e)} S^{(e)} \right) \eta^{(b)(c)} \left(\omega_{(a)(c)(o)} - \omega_{(c)(a)(o)} \right) \right]. \tag{43}
\end{aligned}$$

The final form of χ'_D is

$$p_{(o)} \xi^{(o)} - \mathcal{E}_m \xi_5 - \mathcal{E}_{(s)5}(\xi) - \mathcal{O}_{(s)5}(\xi) \approx 0,$$

$$\begin{aligned}
\mathcal{E}_m &= \sqrt{m^2 c^2 - \eta^{(a)(b)} p_{(a)} p_{(b)}} + \frac{p_{(a)} \eta^{(a)(b)} \omega_{(b)(c)(d)} \epsilon^{(c)(d)}_{(e)} S^{(e)}}{2 \sqrt{m^2 c^2 - \eta^{(a)(b)} p_{(a)} p_{(b)}}}, \\
\mathcal{E}_{(s)5}(\xi) &= \frac{1}{2} \omega_{(o)(b)(c)} \epsilon^{(b)(c)}_{(d)} S^{(d)} \xi^{(o)} + \frac{p_{(o)} \overset{o}{\omega}_{(a)(o)(b)} \epsilon^{(a)(b)}_{(c)} S^{(c)}}{\sqrt{m^2 c^2 + \alpha_p}} \xi_5 + \\
&+ \frac{mc - \sqrt{m^2 c^2 + \alpha_p}}{\alpha_p \sqrt{m^2 c^2 + \alpha_p}} p_{(m)} \left(p_{(o)} \overset{o}{\omega}_{(o)(o)(a)} + \right. \\
&+ \left. p_{(b)} \eta^{(b)(c)} \overset{o}{\omega}_{(a)(o)(c)} \right) \epsilon^{(a)(m)}_{(n)} S^{(n)} \xi^{(o)},
\end{aligned}$$

$$\mathcal{O}_{(s)5}(\xi) = O\left(\frac{1}{mc}\right). \quad (44)$$

The mass shell constraint is $\chi' = -i\{\chi'_D, \chi'_D\}$ and the new Dirac Hamiltonian is $H'_D = \frac{dS}{d\tau} + \lambda(\tau)\chi' + \lambda_\xi(\tau)\chi'_D$.

Once these calculations will be completed and the quantization of the pseudo-classical FW transformation will be done, we will have a first control on the coupling of positive-energy $(\frac{1}{2}, 0)$ 2-spinors to an arbitrary external gravitational field in the framework of tetrad gravity. Since due to Ref.[10, 12] it is clear how to disentangle the inertial effects (gauge variables) from the genuine tidal effects (the two pairs of canonically conjugate Dirac observables, becoming the spin-2 degrees of freedom of the weak field approximation), it will be possible to study the problem of the spin-rotation couplings in gravity [1, 2, 3, 4] in the general case of no Killing vectors and the reliability of the use of the equivalence principle in this area.

Till now this topic has been treated only in Schwarzschild, Kerr and Post-Newtonian spherically symmetric space-times (they are enough for applications inside the solar system) but only in the 1+3 point of view. The best results for the quantum FW transformation in this special class of space-times have been obtained in Refs.[60], after the block-diagonalization of the Dirac Hamiltonian with the Eriksen-Korlsrud method (inequivalent to the FW transformation already in the free case) done in Ref.[61], which had shown the presence of a dipole spin-gravity coupling absent in Minkowski accelerated frames [37].

The main results of Ref.[60] in a weak spherically symmetric gravitational field is the absence of the precession of spin of fermions at rest (the spin rotation is the de Sitter one like for a classical gyroscope) and the validity of the equivalence principle understood as minimal coupling of fermions to gravity. However different observables have different behaviors: i) the helicity of a massive Dirac particle has the same evolution in these gravitational fields and in accelerated frames (in the 1+3 point of view); ii) the spin and the momentum (defining the helicity motion) rotate in the same direction but with different angular velocities (in the semiclassical limit $\omega_{spin} - \omega_{momentum} = \frac{m}{p^2} \vec{g} \times \vec{p}$ with $\vec{g} = -\frac{GM}{r^3} \vec{r}$), which differ from the ones in accelerated frames by kinematical corrections (the spin precession is 3 times bigger).

In the second paper of Ref.[60], it is said that the Newtonian equivalence principle (equality of inertial and gravitational masses) and its Post-Newtonian version (absence of gravitational analogs of electric dipole and anomalous magnetic moments) are till now well tested. Then there is the following version of the Post-Newtonian equivalence principle: the equality of the frequencies of precession of quantum (spin) and classical (orbital) angular momenta and the preservation of the helicity of a Dirac particle in a gravito-magnetic field. In this paper [60] and in Ref. [3] there are proposals of future experiments to test which point of view is the more suitable to describe this important area of gravitational effects.

3.3.2 The Dirac Equation in Tetrad Gravity

Finally the study started [62] of tetrad gravity coupled to the Dirac field to find the full set of constraints both for gravity and for the Dirac field as matter to extend the results of Ref.[36] to curved space-times.

When this work will be completed, one will be able to study the FW transformation in general space-times along the lines of Ref.[60] for the special class of space-times with Killing symmetries without relying on the pseudo-classical approximation. This will give the most general answer to the problem of spin-gravity couplings both in the solar system and in astrophysical contexts. In particular one will be able to see whether the results of the 1+3 point of view about massless fermions (neutrinos) [63] will be confirmed.

4 Conclusions

In this paper we have reviewed and discussed the most important aspects of the problem of the spin-rotation couplings, passing from special to general relativity as well as from classical to semiclassical and quantum physics. The present status of the research and an original presentation of material otherwise scattered into a large number of papers is discussed. All the theoretical expectations and difficulties concerning the existence of a spin-rotation coupling, as foreseen by Mashhoon long ago, are framed both into the “1+3” point of view and into a more general context involving “3+1” space-time splitting techniques.

In this way we have been looking at the status of the art as well as at the open issues and the way to proceed. Due to the lack of experimental data, it is not yet clear which of the two points of view is more suitable to describe physics on the Earth and inside the solar system. At the classical level many of the complications introduced by the 3+1 view may be negligible, but at the quantum level the presence of time-dependent unitary transformations may change the situation modifying the coupling present in some 1+3 approaches.

In conclusion, it would be important to understand better the transition from the “1+3” view to the “3+1” one. A first tool has been introduced in Ref.[9], where it is shown that any non-surface-forming congruence of time-like observers with unit 4-velocity field u^μ of the 1+3 point of view can be reinterpreted as a congruence in which $u^\mu(z(\tau, \sigma^r)) = \partial z^\mu(\tau, \sigma^r) / \partial \tau / \sqrt{g_{\tau\tau}(\tau, \sigma^r)}$ for some Møller admissible embedding $z^\mu(\tau, \sigma^r)$ defining a good clock synchronization convention on instantaneous 3-spaces (ϵ_A^μ is an arbitrary orthonormal tetrad and $C(\tau)$ is a function sufficiently small so that Møller conditions are satisfied)

$$\begin{aligned} z^\mu(\tau, \vec{\sigma}) &= x^\mu(\tau) + \epsilon_r^\mu \sigma^r + \int_0^\tau d\tau_1 C(\tau_1) \epsilon_{\tau\nu} \left[u^\nu(\tau_1, \vec{\sigma}) u^\mu(\tau_1, \vec{\sigma}) \right. \\ &\quad \left. - u^\nu(\tau, \vec{0}) u^\mu(\tau, \vec{0}) \right]. \end{aligned} \quad (45)$$

As a consequence, given any congruence associated with nonzero vorticity, we can find admissible “3+1” splittings of Minkowski space-time, with the space-

like simultaneity leaves not in general not orthogonal to the reference world line $x^\mu(\tau)$ chosen as the origin, which allow to define genuine instantaneous 3-spaces with synchronized clocks.

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